Lecture 2 (1/5/22)

Schwarz Lemma. Let $\mathbb{D} = \{|z|<1\}$ and f analytic in $\mathbb{D}$. Assume $|f(z)| \le 1$ in $\mathbb{D}$ and $f(0) = 0$. Then,

(i) $|f(z)| \le |z|$ and $|f'(0)| \le 1$.

(ii) If $|f(z_0)| = |z_0|$ for $z_0 \ne 0$ or $|f'(0)| = 1$, then $f(z) = cz$, $|c| = 1$.

Pf. Since $f(0)=0$, the function $g(z) = \frac{f(z)}{z}$ has a removable sing. at $z=0$ ($f(0)=0 \Rightarrow f(z) = z\,h(z)$).

We may extend g to be analytic in $\mathbb{D}$ w/ $g(0) = \lim_{z \to 0} g(z) = f'(0)$. Since $|f(z)| \le 1$ in $\mathbb{D}$, $\limsup_{z \to \zeta \in \partial\mathbb{D} = \{|z|=1\}} |g(z)| \le 1$.

By MMT-III, $|g(z)| \le 1$ in $\mathbb{D} \Rightarrow$

$|f(z)| \le |z|$ and $|g(0)| = |f'(0)| \le 1$.

If $\exists\, z_0 \in \mathbb{D} \setminus \{0\}$ s.t. $|f(z_0)| = |z_0|$ or $|f'(0)| = 1$, then $|g(z_0)| = 1$ or $|g(0)| = 1$, which by MMT-I $\Rightarrow g(z) = c = \text{const.}$ and of course $|c| = 1$. This $\Rightarrow$ $f(z) = cz$. $\boxtimes$

<u>An application</u>. We shall use the SL to describe the automorphisms (<u>analytic bijections</u>, or <u>biholomorphisms</u>) of $\mathbb{D}$.

<u>Recall</u>. For $a \in \mathbb{D}$, $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$.

We proved in the fall:

<u>Prop1.</u> For $a \in \mathbb{D}$, φ_a is an automorphism of $\mathbb{D}$.

Pf. It is easy to see that $|\varphi_a(z)|=1$ for $|z|=1$ (DIY). By MMT, $|\varphi_a(z)|<1$ in $\mathbb{D}$ and $\varphi_a(z)=0$ has precisely one root (at $z=a$). Moreover, since $|\varphi_a|=1$ on $\partial\mathbb{D}$, $\sigma=\varphi_a\circ\gamma$ where $\gamma(t)=e^{it}$ for $t\in[0,2\pi]$ is a closed curve traversing $\partial\mathbb{D}$ and hence $n(\sigma,w)=n(\sigma,0)=1$ for all $w\in\mathbb{D}$. By Arg. Princ., $\varphi_a(z)=w$ has precisely one root for all $w\in\mathbb{D}$. The conclusion of Prop 1 has been proved. ☒

Def. (2) For $G\subseteq\mathbb{C}$, $\mathrm{Aut}(G)$ denotes the group of automorphisms of G.

Thm 1. Any $\varphi \in \mathrm{Aut}(\mathbb{D})$ is of the form $\varphi = c\varphi_a$, $|c|=1$, $a \in \mathbb{D}$.

By Prop 1, $c\varphi_a \in \mathrm{Aut}(\mathbb{D})$ so it suffices to prove the converse. For this, we shall use the following consequence of SL.

Thm 2. Let f be analytic in $\mathbb{D}$, $|f(z)| \le 1$, and assume $f(a) = \alpha$, for some $a, \alpha \in \mathbb{D}$. Then,

$$|f'(a)| \le \frac{1-|\alpha|^2}{1-|a|^2}$$

$|c|=1$ const.

with "=" $\iff$ $f(z) = \varphi_{-\alpha}(c\varphi_a(z))$.

Pf. Let $g = \varphi_\alpha \circ f \circ \varphi_{-a}$. By Prop 1, g is anal. in $\mathbb{D}$, $|g| \le 1$, and

$$g(0) = \varphi_\alpha(f(\varphi_{-a}(0))) = \varphi_\alpha(f(a)) = \varphi_\alpha(\alpha) = 0.$$

Thus, by SL, $|g'(0)| \leq 1$. But,

$$g'(0) = \varphi_\alpha'(\alpha) \cdot f'(a) \cdot \varphi_{-a}'(0)$$

$$= f'(a) \cdot \frac{1-|a|^2}{1-|\alpha|^2} \qquad \left(\varphi_b'(z) = \frac{1-|b|^2}{(1-\bar{b}z)^2} \right)$$

$$\Rightarrow |f'(a)| \leq \frac{1-|\alpha|^2}{1-|a|^2} \text{ as desired.}$$

Now, "=" holds $\Leftrightarrow |g'(0)| = 1 \Rightarrow$ $g(z) = cz$. Using $\varphi_b^{-1} = \varphi_{-b}$ (easy to check), one obtains $f(z) = \varphi_{-\alpha}(c\varphi_a(z))$ as desired.

$\boxtimes$

<u>Pf of Thm 1.</u> Let $\psi \in Aut(\mathbb{D})$. Pick $a, \alpha \in \mathbb{D}$ s.t. $\psi(a) = \alpha$. By Thm 2,

$$|\psi'(a)| \leq \frac{1-|\alpha|^2}{1-|a|^2}. \qquad (1)$$

But Thm 2 also applies to ψ^{-1} w/ a, α reversed. $\Rightarrow$

$$|(\psi^{-1})'(\alpha)| \leq \frac{1-|a|^2}{1-|\alpha|^2}. \qquad (2)$$

Now, $(\psi^{-1})'(\alpha) = \dfrac{1}{\psi'(a)}$ since

$\psi^{-1} \circ \psi = Id$. Hence, (1) & (2) $\Rightarrow$ "=" in (1) $\Rightarrow$

$$\psi(z) = \varphi_{-\alpha}(c\varphi_a(z)).$$

Now, since $\psi \in Aut(\mathbb{D})$, we may pick $\alpha = 0$ and $a \in \mathbb{D}$ to be single zero of ψ.

$\Rightarrow \quad \psi = c\varphi_a$ as desired. $\boxtimes$

Typical application of SL + Thm 2.

Ex ① Let f be analytic in $\mathbb{D}$, $|f| \le 1$, and $f(0) = \alpha \neq 0$. Then, f has no no zeros in the disk $|z| < |\alpha|$. How about on $|z| = |\alpha|$?

<u>Solution</u>. First, note that if $|\alpha| = 1$, then f is constant $= \alpha$ by MMT-I. Thus, WLOG, may assume $|\alpha| < 1$. Note that φ_α is a Möbius w/ a pole at $z = \frac{1}{\bar{\alpha}} \notin \overline{\mathbb{D}}$ and $|\varphi_\alpha| \le 1$ in $\overline{\mathbb{D}}$. Thus, $g = \varphi_\alpha \circ f$ is analytic in $\mathbb{D}$, $|g| \le 1$, and $g(0) = \varphi_\alpha(\alpha) = 0$. By SL, $|g(z)| \le |z|$ w/ "=" iff $g = cz$, $|c| = 1$. Thus,

$$\left| \frac{f(z) - \alpha}{1 - \bar{\alpha} f(z)} \right| \le |z| \quad \text{and hence}$$

$f(z) = 0 \Rightarrow |z| \ge |\alpha|$ as claimed.

If $f(z)=0$ on $|z|=|\alpha|$, then we have "$=$" in SL at that point and $z\neq 0$. Thus, SL again $\Rightarrow$ $g(z)=cz$, or $\varphi_\alpha \circ f = c\,\mathrm{Id}$

$\Rightarrow$ $f(z)=\varphi_{-\alpha}(cz)$.